

Negation and partial axiomatizations of dependence and independence logic revisited

Fan Yang

Department of Values, Technology and Innovation, Delft University of Technology, Jaffalaan 5, 2628 BX Delft, The Netherlands

Abstract

In this paper, we axiomatize the negatable consequences in dependence and independence logic by extending the natural deduction systems of the logics given in [21] and [11]. We give a characterization for negatable formulas in independence logic and negatable sentences in dependence logic, and identify an interesting class of formulas that are negatable in independence logic. Dependence and independence atoms, first-order formulas belong to this class. We also give explicit derivations for Armstrong's Axioms and the Geiger-Paz-Pearl axioms of dependence and independence atoms in our extended system of independence logic.

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1. Introduction

Negation and partial axiomatizations of dependence and independence logic have been studied in the literature. In this paper, we take a new look at these topics.

Dependence logic was introduced by Väänänen [24] as a development of *Henkin quantifier* [12] and *independence-friendly logic* [13]. Recently, Grädel and Väänänen [10] defined a variant of dependence logic, called *independence logic*. The two logics add to first-order logic new types of atomic formulas $=(\vec{x}, y)$ and $\vec{x} \perp_{\vec{z}} \vec{y}$, called *dependence atom* and *independence atom*, to explicitly specify the dependence and independence relations between variables. Intuitively, $=(\vec{x}, y)$ states that “the value of y is completely determined by the values of the variables in the tuple \vec{x} ”, and $\vec{x} \perp_{\vec{z}} \vec{y}$ states that “given the values of the variables \vec{z} , the values of \vec{x} and the values of \vec{y} are completely independent of each other”. These properties cannot be meaningfully manifested in *single* assignments of the variables. Therefore unlike in the case of the usual Tarskian semantics, formulas of dependence and independence logic are evaluated on *sets* of assignments (called *teams*) instead. This semantics is called *team semantics* and was introduced by Hodges [14, 15].

Email address: fan.yang.c@gmail.com (Fan Yang)

Dependence and independence logic are known to have the same expressive power as existential second-order logic Σ_1^1 (see [19] and [6]). This fact has two negative consequences: The logics are not closed under classical negation and are not axiomatizable. The aim of this paper is to shed some new light on these problems.

Regarding the first problem, “negation”, which is usually a desirable connective for a logic, turns out to be a tricky connective in the context of team semantics. The negation that dependence and independence logic inherit from first-order logic (denoted by \neg) is a type of “syntactic negation”, in the sense that in order to compute the meaning of the formula $\neg\phi$, the negation \neg has to be brought to the very front of atomic formulas by applying De Morgan’s laws and the double negation law. It was proved that this negation \neg is actually not a semantic operator [20], meaning that that ϕ and ψ are semantically equivalent does not necessarily imply that $\neg\phi$ and $\neg\psi$ are semantically equivalent. The *classical (contradictory) negation* (denoted by \sim in the literature), on the other hand, is a semantic operator. Since the Σ_1^1 fragment of second-order logic is not closed under classical negation, neither dependence nor independence logic is closed under classical negation. Dependence logic extended with the classical negation \sim is called *team logic* in the literature, and it has the same expressive power as full second-order logic (see [24] and [18]).

Since every formula of dependence and independence logic is satisfied on the empty team, the classical contradictory negation $\sim\phi$ of any formula will not be satisfied on the empty team, implying that $\sim\phi$ cannot possibly be definable in dependence or independence logic for any single formula ϕ . This technical subtlety makes the classical contradictory negation \sim less interesting. In this paper, we will, instead, consider the *weak classical negation*, denoted by $\tilde{\sim}$, which behaves exactly as the classical negation except that on the empty team $\tilde{\sim}\phi$ is always satisfied. We will give a characterization for negatable formulas in independence logic and negatable sentences in dependence logic by generalizing an argument in [24]. We also identify an interesting class of formulas that are negatable in independence logic. First-order formulas, dependence and independence atoms belong to this class. Formulas of this class are closely related to the *dependency notions* considered in [7] and the *generalized dependence atoms* studied in [22] and [17].

As for the axiomatization problem, since Σ_1^1 is not axiomatizable, dependence and independence logic cannot possibly be axiomatized in full. Nevertheless, [21] and [11] defined natural deduction systems for the logics such that the equivalence

$$\Gamma \models \phi \iff \Gamma \vdash \phi \tag{1}$$

holds if Γ is a set of sentences of dependence or independence logic and ϕ is a first-order sentence. It was left open whether these partial axiomatizations can be generalized such that the above equivalence (1) holds if Γ is a set of formulas (that possibly contain free variables) and ϕ is a (possibly open) first-order formula. Kontinen [16] gave such a generalization by expanding the signature with an extra relation symbol so as to interpret the teams associated with the free variables. In this paper, we will generalize the partial axiomatization results in [21] and [11] via a different approach, an approach that makes use of the weak classical negation. We will define extensions of the systems given in [21] and [11] such that the equivalence (1) holds if Γ is a set of formulas and ϕ is a formula that is negatable in the logics.

This paper is organized as follows. In Section 2 we recall the basics of dependence and independence logic. Section 3 gives the characterization of negatable formulas or sentences in the logics. In Section 4 we extend the natural deduction systems of dependence and independence logic in [21] and [11] to axiomatize negatable consequences in the logics. Section 5 discusses a class of negatable formulas in independence logic, of which dependence and independence atoms are members. In Section 6 we illustrate the extended systems by deriving Armstrong's Axioms and the Geiger-Paz-Pearl axioms of dependence and independence atoms in the extended system of independence logic. We finish by making some concluding remarks in Section 7.

2. Preliminaries

Let us start by recalling the syntax and semantics (i.e. team semantics) of dependence and independence logic.

Although team semantics is intended for extensions of first-order logic obtained by adding dependence or independence atoms, for the sake of comparison we will now introduce the team semantics for first-order logic too. First-order atomic formulas α for a given signature \mathcal{L} are defined as usual. Well-formed formulas of first-order logic, also called *first-order formulas*, (in *negation normal form*) are defined by the following grammar:

$$\phi ::= \alpha \mid \neg\alpha \mid \perp \mid \phi \wedge \phi \mid \phi \vee \phi \mid \exists x\phi \mid \forall x\phi$$

Formulas will be evaluated on the usual first-order models over an appropriate signature \mathcal{L} . We will use the same notation M for both a model and its domain, and assume that M has at least two elements. Let R be a fresh k -ary relation symbol and R^M a k -ary relation on M . We write $\mathcal{L}(R)$ for the expanded signature and (M, R^M) denotes the $\mathcal{L}(R)$ -expansion of M in which the relation symbol R is interpreted as R^M . We write $\phi(R)$ to emphasize that the relation symbol R occurs in the formula ϕ .

Definition 2.1. *Let M be a model and V a set of first-order variables. A team X of M over V is a set of assignments of M over V , i.e., a set of functions $s : V \rightarrow M$. The set V is called the domain of X , denoted by $\text{dom}(X)$.*

There is one and only one assignment of M over the empty domain, namely the empty assignment \emptyset . The singleton of the empty assignment $\{\emptyset\}$ is a team of M , and the empty set \emptyset is a team of M over any domain.

Let s be an assignment of M over V and $a \in M$. We write $s(a/x)$ for the assignment of M over $V \cup \{x\}$ defined as $s(a/x)(x) = a$ and $s(a/x)(y) = s(y)$ for all $y \in V \setminus \{x\}$. For any set $N \subseteq M$ and any function $F : X \rightarrow \wp(M) \setminus \{\emptyset\}$, define

$$X(N/x) = \{s(a/x) : a \in N, s \in X\} \text{ and } X[F/x] = \{s(a/x) : s \in X \text{ and } a \in F(s)\}$$

We write \vec{x} for a sequence x_1, \dots, x_n of variables and the length n will always be clear from the context or does not matter; similarly for a sequence \vec{F} of functions and a sequence \vec{s} of assignments. A team $X(M/x_1) \dots (M/x_n)$ will sometimes be abbreviated as $X(M/\vec{x})$, and $X[F_1/x_1] \dots [F_n/x_n]$ as $X[F_1/x_1, \dots, F_n/x_n]$ or $X[\vec{F}/\vec{x}]$.

We now define the team semantics for first-order formulas. Note that our version of the team semantics for disjunction and existential quantifier is known as the *lax semantics* in the literature.

Definition 2.2. Define inductively the notion of a first-order formula ϕ being satisfied on a model M and a team X , denoted by $M \models_X \phi$, as follows:

- $M \models_X \alpha$ with α a first-order atomic formula iff for all $s \in X$, $M \models_s \alpha$ in the usual sense
- $M \models_X \neg\alpha$ with α a first-order atomic formula iff for all $s \in X$, $M \models_s \neg\alpha$ in the usual sense
- $M \models_X \perp$ iff $X = \emptyset$
- $M \models_X \phi \wedge \psi$ iff $M \models_X \phi$ and $M \models_X \psi$
- $M \models_X \phi \vee \psi$ iff there exist $Y, Z \subseteq X$ with $X = Y \cup Z$ such that $M \models_Y \phi$ and $M \models_Z \psi$
- $M \models_X \exists x\phi$ iff $M \models_{X[F/x]} \phi$ for some function $F : X \rightarrow \wp(M) \setminus \{\emptyset\}$
- $M \models_X \forall x\phi$ iff $M \models_{X(M/x)} \phi$

A routine inductive proof shows that first-order formulas have the downward closure property and the union closure property:

(Downward Closure Property) $M \models_X \phi$ and $Y \subseteq X$ imply $M \models_Y \phi$

(Union Closure Property) $M \models_{X_i} \phi$ for all $i \in I$ implies $M \models_{\bigcup_{i \in I} X_i} \phi$

which combined are equivalent to the flatness property:

(Flatness Property) $M \models_X \phi \iff M \models_{\{s\}} \phi$ for all $s \in X$

It follows easily from the flatness property that the team semantics for first-order formulas coincides with the usual single-assignment semantics in the sense that

$$M \models_{\{s\}} \phi \iff M \models_s \phi \quad (2)$$

holds for any model M , any assignment s and any first-order formula ϕ . If ϕ is a first-order formula, then the string $\neg\phi$, called the *syntactic negation* of ϕ , can be viewed as a first-order formula in negation normal form obtained in the usual way (i.e. by applying De Morgan's laws, the double negation law, etc.), and we write $\phi \rightarrow \psi$ for the formula $\neg\phi \vee \psi$. Since first-order formulas satisfy the Law of Excluded Middle $\phi \vee \neg\phi$ under the usual single-assignment semantics, Expression (2) implies that $M \models_{\{s\}} \phi \vee \neg\phi$ always holds, which, together with the flatness property, implies that $M \models_X \phi \vee \neg\phi$ holds for all teams X and all models M , namely, the Law of Excluded Middle holds for first-order formulas also in the sense of team semantics.

We now turn to dependence and independence logic. Well-formed formulas of independence logic (\mathcal{I}) are defined by the following grammar:

$$\begin{aligned} \phi ::= & \alpha \mid \neg\alpha \mid \perp \mid x_1 \dots x_n \perp_{z_1 \dots z_k} y_1 \dots y_m \mid =(x_1, \dots, x_n, y) \mid x_1 \dots x_n \subseteq y_1 \dots y_n \mid \\ & \phi \wedge \phi \mid \phi \vee \phi \mid \exists x \phi \mid \forall x \phi \end{aligned}$$

where α ranges over first-order atomic formulas. The formulas $=(\vec{x}, y)$, $\vec{x} \perp_{\vec{z}} \vec{y}$ and $\vec{x} \subseteq \vec{y}$ are called *dependence atom*, *independence atom* and *inclusion atom*, respectively. We refer to any of these atoms as *atoms of dependence and independence*. For the convenience of our argument in the paper, the independence logic as defined has a richer syntax than the standard one in the literature, which has the same syntax as first-order logic extended with independence atoms only. The other atoms are definable in the standard independence logic; for a proof, see e.g., [5]. *Dependence logic* (\mathcal{D}), which is a fragment of \mathcal{I} , is defined as first-order logic extended with dependence atoms, and first-order logic extended with inclusion atoms is called *inclusion logic*. In this paper we will only concentrate on dependence logic and independence logic.

The set $\text{Fv}(\phi)$ of free variables of a formula ϕ of \mathcal{I} is defined as usual and we also have the new cases for dependence and independence atoms:

- $\text{Fv}(x_1 \dots x_n \perp_{z_1 \dots z_k} y_1 \dots y_m) = \{x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_k\}$
- $\text{Fv}(=(x_1, \dots, x_n, y)) = \{x_1, \dots, x_n, y\}$
- $\text{Fv}(x_1, \dots, x_n \subseteq y_1, \dots, y_n) = \{x_1, \dots, x_n, y_1, \dots, y_n\}$

We write $\phi(\vec{x})$ to indicate that the free variables occurring in ϕ are among \vec{x} . A formula ϕ is called a *sentence* if it has no free variable.

Definition 2.3. Define inductively the notion of a formula ϕ of \mathcal{I} being satisfied on a model M and a team X , denoted by $M \models_X \phi$. All the cases are identical to those defined in 2.2 and additionally:

- $M \models_X \vec{x} \perp_{\vec{z}} \vec{y}$ iff for all $s, s' \in X$, $s(\vec{z}) = s'(\vec{z})$ implies that there exists $s'' \in X$ such that

$$s''(\vec{z}) = s(\vec{z}) = s'(\vec{z}), s''(\vec{x}) = s(\vec{x}) \text{ and } s''(\vec{y}) = s'(\vec{y}).$$

- $M \models_X =(x, y)$ iff for all $s, s' \in X$, $s(x) = s'(x)$ implies $s(y) = s'(y)$.
- $M \models_X \vec{x} \subseteq \vec{y}$ iff for all $s \in X$, there exists $s' \in X$ such that $s'(\vec{y}) = s(\vec{x})$.

We write $\vec{x} \perp \vec{y}$ for $\vec{x} \perp_{\emptyset} \vec{y}$, and note that the semantic clause for $\vec{x} \perp \vec{y}$ reduces to

- $M \models_X \vec{x} \perp \vec{y}$ iff for all $s, s' \in X$, there exist $s'' \in X$ such that

$$s''(\vec{x}) = s(\vec{x}) \text{ and } s''(\vec{y}) = s'(\vec{y}).$$

A sentence ϕ is said to be *true* in M , written $M \models \phi$, if $M \models_{\{0\}} \phi$. We write $\Gamma \models \psi$ if for any model M and any team X , $M \models_X \phi$ for all $\phi \in \Gamma$ implies $M \models_X \psi$. We also write $\phi \models \psi$ for $\{\phi\} \models \psi$. If $\phi \models \psi$ and $\psi \models \phi$, then we write $\phi \equiv \psi$.

We leave it for the reader to verify that formulas of dependence logic have the downward closure property and formulas of independence logic have the empty team property and the locality property:

(Empty Team Property) $M \models_{\emptyset} \phi$

(Locality Property) If $\{s \upharpoonright \text{Fv}(\phi) \mid s \in X\} = \{s \upharpoonright \text{Fv}(\phi) \mid s \in Y\}^1$, then

$$M \models_X \phi \iff M \models_Y \phi.$$

Recall that the existential second-order logic (Σ_1^1) consists of those formulas that are equivalent to some formulas of the form $\exists R_1 \dots \exists R_k \phi$, where ϕ is a first-order formula. An $\mathcal{L}(R)$ -sentence $\phi(R)$ of Σ_1^1 is said to be *downward monotone* with respect to R if $(M, Q) \models \phi(R)$ and $Q' \subseteq Q$ imply $(M, Q') \models \phi(R)$. It is known that $\phi(R)$ is downward monotone with respect to R if and only if R occurs in $\phi(R)$ only negatively (see e.g., [19]). A team X of M over $\{x_1, \dots, x_n\}$ induces an n -ary relation

$$\text{rel}(X) := \{(s(x_1), \dots, s(x_n)) \mid s \in X\}$$

on M ; conversely, an n -ary relation R on M induces a team

$$X_R := \{(x_1, a_1), \dots, (x_n, a_n) \mid (a_1, \dots, a_n) \in R\}.$$

Theorem 2.4 (see [24], [19] and [6]). **(i)** Every \mathcal{L} -sentence ϕ of \mathcal{D} or \mathcal{I} is equivalent to an \mathcal{L} -sentence τ_ϕ of Σ_1^1 , i.e.,

$$M \models \phi \iff M \models \tau_\phi$$

holds for any model M ; and conversely, every \mathcal{L} -sentence of Σ_1^1 is equivalent to an \mathcal{L} -sentence $\rho(\psi)$ of \mathcal{D} or \mathcal{I} .

(ii) For every \mathcal{L} -formula ϕ of \mathcal{I} , there is an $\mathcal{L}(R)$ -sentence $\tau_\phi(R)$ of Σ_1^1 such that for all models M and all teams X ,

$$M \models_X \phi \iff (M, \text{rel}(X)) \models \tau_\phi(R).$$

If, in particular, ϕ is a formula of \mathcal{D} , then the relation symbol R occurs in the sentence $\tau_\phi(R)$ only negatively.

(iii) For every $\mathcal{L}(R)$ -sentence $\psi(R)$ of Σ_1^1 that is downward monotone with respect to R , there is an \mathcal{L} -formula $\rho(\psi)$ of \mathcal{D} such that for all models M and all teams X ,

$$M \models_X \rho(\psi) \iff (M, \text{rel}(X)) \models \psi(R) \vee \forall \vec{x} \neg R\vec{x}. \quad (3)$$

¹For an assignment $s : V \rightarrow M$ and a set $V' \subseteq V$ of variables, we write $s \upharpoonright V'$ for the restriction of s to the domain V' .

(iv) For every $\mathcal{L}(R)$ -sentence $\psi(R)$ of Σ_1^1 , there is an \mathcal{L} -formula $\rho(\psi)$ of \mathcal{I} such that (3) holds for all models M and all teams X .

In the sequel, we will use the notations τ_ϕ and $\tau_\phi(R)$ to denote the (up to semantic equivalence) unique formulas obtained in the above theorem and refer to them as the Σ_1^1 -translations of the formulas ϕ of \mathcal{D} or \mathcal{I} .

3. First-order formulas and negatable formulas

Formulas of dependence and independence logic can be translated into Σ_1^1 (Theorem 2.4). Therefore in the environment of team semantics a first-order formula ϕ has two identities: It can be viewed either as a formula of \mathcal{D} or \mathcal{I} that is to be evaluated on teams, or as a usual formula of first-order logic that is to be evaluated on single assignments and is possibly (equivalent to) the Σ_1^1 -translation τ_ψ of some formula ψ of \mathcal{D} or \mathcal{I} . With the latter reading of a first-order formula ϕ , for all models M and all assignments s , $M \models_s \neg\phi$ iff $M \not\models_s \phi$ holds. In this sense, the formula $\neg\phi$ can be interpreted as the “classical (contradictory) negation” of ϕ . However, on the team semantics side, unless the team X is a singleton, $M \not\models_X \phi$ is in general not equivalent to $M \models_X \neg\phi$. To express the contradictory negation in the team semantics setting, let us define the *classical negation* \sim and the *weak classical negation* \sim as follows:

- $M \models_X \sim \phi$ iff $M \not\models_X \phi$
- $M \models_X \sim \phi$ iff either $M \not\models_X \phi$ or $X = \emptyset$

Since formulas of dependence and independence logic have the empty team property, the classical negation $\sim \phi$ of any formula ϕ is not definable in the logics and we are therefore not interested in the classical negation \sim in this paper. On the other hand, the weak classical negation $\sim \phi$ can be definable in the logics for some formulas ϕ . We say that a formula ϕ is *negatable* in \mathcal{I} (or \mathcal{D}) if there is a formula ψ of \mathcal{I} (or \mathcal{D}) such that $\sim \phi \equiv \psi$. If a formula ϕ of \mathcal{I} is negatable in \mathcal{I} , we also say that ϕ is a negatable formula in \mathcal{I} or the formula ϕ of \mathcal{I} is negatable; similarly for \mathcal{D} .

For any first-order sentence ϕ , we have $M \not\models_{\{\emptyset\}} \phi$ iff $M \models_{\{\emptyset\}} \neg\phi$ by the Law of Excluded Middle. Thus $\sim \phi \equiv \neg\phi$, meaning that first-order sentences are negatable both in \mathcal{D} and in \mathcal{I} . Next, we prove that negatable formulas in \mathcal{D} are, actually, all flat.

Fact 3.1. *If a formula ϕ of \mathcal{D} is negatable in \mathcal{D} , then it is upward closed (i.e. $M \models_X \phi$ and $\emptyset \neq X \subseteq Y$ imply $M \models_Y \phi$), and thus flat.*

Proof. Suppose ϕ is a formula of \mathcal{D} that is not upward closed. Then, there exist a model M and two teams $X \neq \emptyset$ and $Y \supseteq X$ such that $M \models_X \phi$ and $M \not\models_Y \phi$. But this means that $\sim \phi$ is not downward closed and thus not definable in \mathcal{D} . \square

We will see in the sequel that the above fact does not apply to independence logic. Also note that sentences are always upward closed (since to evaluate a sentence it is sufficient to consider the nonempty team $\{\emptyset\}$ only). Thus, the other direction of the above fact, if true, would imply that all sentences of \mathcal{D} are negatable. But this is not the case, as we will see in the following characterization theorem for negatable sentences in \mathcal{D} and negatable formulas in \mathcal{I} .

Theorem 3.2. (i) *An \mathcal{L} -formula ϕ of \mathcal{I} is negatable in \mathcal{I} if and only if its Σ_1^1 -translation $\tau_\phi(R)$ is equivalent to a first-order sentence.*

(ii) *An \mathcal{L} -sentence ϕ of \mathcal{D} is negatable in \mathcal{D} if and only if its Σ_1^1 -translation τ_ϕ is equivalent to a first-order sentence.*

The above theorem states that negatable formulas in \mathcal{I} are exactly those formulas that have first-order translations, and negatable sentences in \mathcal{D} are exactly those sentences that have first-order translations. Therefore the problem of determining whether a formula of \mathcal{I} or a sentence of \mathcal{D} is negatable reduces to the problem of determining whether a Σ_1^1 -sentence (τ_ϕ) is equivalent to a first-order formula, or whether the second-order quantifiers in a Σ_1^1 -sentence can be eliminated. This problem is known to be undecidable (this follows from e.g., [4]).

Before we give the proof of Theorem 3.2, it is worth making some comments on yet another type of negation, the *intuitionistic negation*. Abramsky and Väänänen [1] introduced the *intuitionistic negation* (denoted \rightarrow) that has the semantics clause:

- $M \models_X \phi \rightarrow \psi$ iff for all $Y \subseteq X$, if $M \models_Y \phi$, then $M \models_Y \psi$.

The *intuitionistic negation* of a formula ϕ is defined (according to the usual convention) as $\phi \rightarrow \perp$ and its semantics clause reduces to:

- $M \models_X \phi \rightarrow \perp$ iff for all nonempty $Y \subseteq X$, if $M \not\models_Y \phi$.

If a formula ϕ of \mathcal{D} is negatable in \mathcal{D} , then both ϕ and $\sim\phi$ are downward closed, and thus $\sim\phi \equiv \phi \rightarrow \perp$. Conversely, $\phi \rightarrow \perp$ is not necessarily equivalent to $\sim\phi$; for instance, $\equiv(x) \rightarrow \perp \equiv \perp \not\equiv \sim\equiv(x)$. Nevertheless, if ϕ is a sentence, then we do have $\sim\phi \equiv \phi \rightarrow \perp$. However, by Theorem 3.2, in this case the formula $\phi \rightarrow \perp$ (that is not in the language of \mathcal{D} or \mathcal{I}) is not necessarily definable in \mathcal{D} or \mathcal{I} . In fact, the second-order translation of the formula $\phi \rightarrow \perp$ is in general a Π_1^1 -sentence, which is, indeed, in general not expressible in \mathcal{D} or \mathcal{I} (see [1] for details).

Let us now turn to the proof of Theorem 3.2. Item (ii) actually follows implicitly from the results in [24], and item (i) can be proved by essentially the argument of Theorem 6.7 in [24]. To proceed, let us first direct our attention to the Σ_1^1 counterpart of dependence and independence logic and prove a general theorem for Σ_1^1 . The proof below is inspired by Theorem 6.7 in [24].

Theorem 3.3. (i) *Let $\phi(R)$ be an $\mathcal{L}(R)$ -formula of Σ_1^1 such that $(M, \emptyset) \models \phi(R)$ for any \mathcal{L} -model M . The formula $\neg\phi \vee \forall \vec{x} \neg R\vec{x}$ belongs to Σ_1^1 if and only if ϕ is equivalent to a first-order formula.*

(ii) *Let ϕ be an \mathcal{L} -formula of Σ_1^1 . The \mathcal{L} -formula $\neg\phi$ belongs to Σ_1^1 if and only if ϕ is equivalent to a first-order formula.*

Proof. (i) It suffices to prove the direction “ \implies ”. Suppose both ϕ and $\neg\phi \vee \forall \vec{x} \neg R\vec{x}$ belong to Σ_1^1 . We may assume without loss of generality that $\phi \equiv \exists S_1 \dots \exists S_k \psi$ and $(\neg\phi \vee \forall \vec{x} \neg R\vec{x}) \equiv \exists T_1 \dots \exists T_m \chi$ for some first-order formulas ψ and χ , and the relation variables

$S_1, \dots, S_k, T_1, \dots, T_m$ are pairwise distinct. Assume also that $\phi(R)$ and $\exists S_1 \dots \exists S_k \psi$ are $\mathcal{L}_1(R)$ -formulas, and $\neg\phi(R) \vee \forall \vec{x} \neg R\vec{x}$ and $\exists T_1 \dots \exists T_m \chi$ are $\mathcal{L}_2(R)$ -formulas.

Claim 1: $\psi \models \neg\chi \vee \forall \vec{x} \neg R\vec{x}$.

Proof of Claim 1. Put $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2 \cup \{R, S_1, \dots, S_k, T_1, \dots, T_m\}$. For any \mathcal{L} -model M such that $M \models \psi$, we have $M \models \exists S_1 \dots \exists S_k \psi$. If $R^M = \emptyset$, then $M \models \forall \vec{x} \neg R\vec{x}$, thereby $M \models \neg\chi \vee \forall \vec{x} \neg R\vec{x}$. If $R^M \neq \emptyset$, then we have $M \models \neg\forall \vec{x} \neg R\vec{x}$. By the assumption, we also have $M \models \phi$. Thus, we derive

$$\begin{aligned} M \models \neg\neg\phi \wedge \neg\forall \vec{x} \neg R\vec{x} &\implies M \models \neg(\neg\phi \vee \forall \vec{x} \neg R\vec{x}) \implies M \models \neg\exists T_1 \dots \exists T_m \chi \\ &\implies M \models \forall T_1 \dots \forall T_m \neg\chi \implies M \models \neg\chi \\ &\implies M \models \neg\chi \vee \forall \vec{x} \neg R\vec{x} \end{aligned}$$

as required. \dashv

Now, by Craig's Interpolation Theorem of first-order logic, there exists a first-order $\mathcal{L}_1(R) \cap \mathcal{L}_2(R)$ -formula θ such that $\psi \models \theta$ and $\theta \models \neg\chi \vee \forall \vec{x} \neg R\vec{x}$.

Claim 2: $\phi \equiv \theta$.

Proof of Claim 2. For any $\mathcal{L}_1(R)$ -model M , if $M \models \phi$, then $(M, S_1^M, \dots, S_k^M) \models \psi$ for some relations S_1^M, \dots, S_k^M on M . Hence, $M \models \theta$.

Conversely, for any $\mathcal{L}_1(R)$ -model M such that $M \not\models \phi$, we have $R^M \neq \emptyset$ and $M \models \neg\phi \vee \forall \vec{x} \neg R\vec{x}$. The latter implies $(M, T_1^M, \dots, T_m^M) \models \chi$ for some relations T_1^M, \dots, T_m^M on M . It then follows that $(M, T_1^M, \dots, T_m^M) \not\models \neg\chi \vee \forall \vec{x} \neg R\vec{x}$. Hence, $M \not\models \theta$. \dashv

(ii) The nontrivial direction “ \implies ” follows from a similar and simplified argument. Instead of proving Claim 1 as above, one proves $\psi \models \neg\chi$. \square

Now, we are ready to give the proof of Theorem 3.2.

Proof of Theorem 3.2. (i) Let ϕ be an \mathcal{L} -formula of \mathcal{I} . By Theorem 2.4(ii) there exists an $\mathcal{L}(R)$ -sentence $\tau_\phi(R)$ of Σ_1^1 such that for any model M and any team X ,

$$M \models_X \sim\phi \iff M \not\models_X \phi \text{ or } X = \emptyset \iff (M, \text{rel}(X)) \models \neg\tau_\phi(R) \vee \forall \vec{x} \neg R\vec{x}. \quad (4)$$

Now, to prove the direction “ \Leftarrow ”, assume that $\tau_\phi(R)$ is equivalent to a first-order sentence. Then, the sentence $\neg\tau_\phi(R)$ is also equivalent to a first-order sentence, and thus by Theorem 2.4(iv) there exists a formula $\rho(\neg\tau_\phi)$ of \mathcal{I} such that for all \mathcal{L} -models M and all teams X ,

$$M \models_X \rho(\neg\tau_\phi) \iff (M, \text{rel}(X)) \models \neg\tau_\phi(R) \vee \forall \vec{x} \neg R\vec{x}.$$

It then follows from (4) that $\rho(\neg\tau_\phi) \equiv \sim\phi$.

Finally, to prove the direction “ \implies ”, assume that $\sim\phi \equiv \psi$ for some formula ψ of \mathcal{I} . By Theorem 2.4(ii) there exists an $\mathcal{L}(R)$ -sentence $\tau_\psi(R)$ of Σ_1^1 such that for all models M and all teams X ,

$$M \models_X \psi \iff (M, \text{rel}(X)) \models \tau_\psi(R).$$

By (4), $\tau_\psi(R) \equiv \neg\tau_\phi(R) \vee \forall \vec{x} \neg R\vec{x}$ and thereby the formula $\neg\tau_\phi(R) \vee \forall \vec{x} \neg R\vec{x}$ belongs to Σ_1^1 . For any model M , since $M \models_\emptyset \phi$, we have $(M, \emptyset) \models \tau_\phi(R)$. Then, by Theorem 3.3(i), we conclude that $\tau_\phi(R)$ is equivalent to a first-order formula.

(ii) This item is proved by a similar argument that makes use of Theorem 2.4(i) and Theorem 3.3(ii). \square

4. Axiomatizing negatable consequences in dependence and independence logic

Dependence and independence logic are not axiomatizable, meaning that the consequence relation $\Gamma \models \phi$ cannot be effectively axiomatized. Nevertheless, if we restrict $\Gamma \cup \{\phi\}$ to a set of sentences and ϕ to a first-order sentence, the consequence relation $\Gamma \models \phi$ is axiomatizable and explicit axiomatizations for \mathcal{D} and \mathcal{I} are given in [21] and [11]. Throughout this section, let L denote one of the logics of \mathcal{D} and \mathcal{I} , and \vdash_{L} denote the syntactic consequence relation associated with the deduction system of L defined in [21] or in [11].

Theorem 4.1 (see [21] and [11]). *Let Γ be a set of sentences of L , and ϕ a first-order sentence. We have $\Gamma \models \phi \iff \Gamma \vdash_{\mathsf{L}} \phi$. In particular, $\Gamma \models \perp \iff \Gamma \vdash_{\mathsf{L}} \perp$.*

Kontinen [16] generalized the above axiomatization result to cover also the case when $\Gamma \cup \{\phi\}$ is a set of formulas (that possibly contain free variables) by adding a new relation symbol to interpret the teams. In this section, we will generalize Theorem 4.1 without expanding the signature to cover the case when $\Gamma \cup \{\phi\}$ is a set of formulas (that possibly contain free variables) and ϕ is negatable.

First, note that by applying the existential quantifier introduction and elimination rule introduced in [21] and [11]:

Existential quantifier introduction	Existential quantifier elimination
$\frac{\phi(t/x)}{\exists x\phi} \exists\text{I}$	$\frac{\begin{array}{c} [\phi] \\ D_1 \quad D_2 \\ \exists x\phi \quad \psi \end{array}}{\psi} (*) \exists\text{E}$
(*) where x does not occur freely in ψ or in any formula in the undischarged assumptions in the derivation D_2	

under certain constraint the (possibly open) formula ψ in the entailment $\Delta, \psi \vdash_{\mathsf{L}} \theta$ can be turned into a sentence without affecting the entailment relation, as in the lemma below.

Lemma 4.2. *Let $\Delta \cup \{\chi, \theta\}$ be a set of formulas of L . Let $V = \{x_1, \dots, x_n\} \supseteq \text{Fv}(\chi)$ and $\text{Fv}(\Delta) = \bigcup_{\delta \in \Delta} \text{Fv}(\delta)$. Suppose that $V \cap \text{Fv}(\Delta) = \emptyset$ and $V \cap \text{Fv}(\theta) = \emptyset$. We have $\Delta, \chi \vdash_{\mathsf{L}} \theta \iff \Delta, \exists x_1 \dots \exists x_n \chi \vdash_{\mathsf{L}} \theta$.*

Proof. Apply the rules $\exists\text{I}$ and $\exists\text{E}$. \square

To understand why Theorem 4.1 can be generalized, let us consider a set $\Gamma \cup \{\phi\}$ of formulas of \mathcal{L} . Since Σ_1^1 admits the Compactness Theorem, we may assume that Γ is a finite set. We shall add to the deduction system of \mathcal{L} the usual (sound) rules to guarantee that $\Gamma \vdash \phi$ follows from $\Gamma, \sim\phi \vdash \perp$. The latter, by 4.2, is equivalent to $\exists \vec{x}(\bigwedge \Gamma \wedge \sim\phi) \vdash \perp$, where $\text{Fv}(\bigwedge \Gamma \wedge \sim\phi) = \{x_1, \dots, x_n\}$. Then, the Completeness Theorem can be restated as $\exists \vec{x}(\bigwedge \Gamma \wedge \sim\phi) \not\vdash \perp \implies \exists \vec{x}(\bigwedge \Gamma \wedge \sim\phi) \not\models \perp$. Assuming that $\exists \vec{x}(\bigwedge \Gamma \wedge \sim\phi)$ is deductively consistent, the problem reduces to the problem of constructing a model for the sentence $\exists \vec{x}(\bigwedge \Gamma \wedge \sim\phi)$. If $\sim\phi$ is definable in \mathcal{L} , then the problem further reduces to the problem of constructing a model for the Σ_1^1 sentence $\tau_{\exists \vec{x}(\bigwedge \Gamma \wedge \sim\phi)}$, which can be transformed to $\exists \vec{R}\theta$ for some first-order sentence θ . Finding a counter-model for $\exists \vec{R}\theta$ is the same as finding a counter-model for the first-order formula θ . This argument shows that via the trick of weak classical negation Theorem 4.1 can, in principle, be generalized. Note that if Γ is a set of sentences and ϕ is a first-order sentence, then $\neg\phi \equiv \sim\phi$ and the foregoing argument reduces to the argument given in [21].

Let us now make this idea precise. Given the Completeness Theorems in [21] and [11], it suffices to extend the natural deduction systems of [21] and [11] by adding a (usual) rule for negation below to ensure that $\Gamma \vdash \phi$ follows from $\Gamma, \sim\phi \vdash \perp$, where $\sim\phi$ denotes the formula of \mathcal{L} expressing the weak negation of ϕ .

NEW RULE

Weak classical negation elimination
$\begin{array}{c} [\sim\phi] \\ \vdots \\ \frac{\perp}{\phi} \sim E \end{array}$

Let $\vdash_{\mathcal{L}}^*$ denote the syntactic consequence relation associated with the system of \mathcal{L} extended with the rule $\sim E$. We now prove the Soundness and Completeness Theorem for this extended system.

Theorem 4.3. *Let $\Gamma \cup \{\phi\}$ be a set of formulas of \mathcal{L} such that ϕ is negatable in \mathcal{L} . We have $\Gamma \models \phi \iff \Gamma \vdash_{\mathcal{L}}^* \phi$.*

Proof. “ \Leftarrow ”: The soundness of the rules of the systems of \mathcal{L} follows from [21] and [11], and the new rule $\sim E$ is clearly sound.

“ \Rightarrow ”: Since \mathcal{L} is compact, without loss of generality we may assume that Γ is finite. By 4.2 and the Completeness Theorem of \mathcal{L} (Theorem 4.1), we derive

$$\begin{aligned} \Gamma \models \phi &\implies \Gamma, \sim\phi \models \perp \implies \exists \vec{x}(\bigwedge \Gamma \wedge \sim\phi) \models \perp \\ &\implies \exists \vec{x}(\bigwedge \Gamma \wedge \sim\phi) \vdash_{\mathcal{L}} \perp \implies \Gamma, \sim\phi \vdash_{\mathcal{L}} \perp \implies \Gamma \vdash_{\mathcal{L}}^* \phi \end{aligned} \quad (5)$$

by applying the rules $\sim E$. □

If $\phi(\vec{x})$ is first-order formula, then one can define its Σ_1^1 -translation as the first-order sentence $\tau_\phi(R) = \forall \vec{x}(R(\vec{x}) \rightarrow \phi(\vec{x}))$. Thus, by Theorem 3.2, ϕ is negatable in \mathcal{I} , and also in \mathcal{D} in case ϕ is a sentence. This shows that our Theorem 4.3 is indeed a generalization of Theorem 4.1 and also of [16] for \mathcal{I} .

A key issue in the application of the extended system is the issue of computing the weak negation of formulas in \mathcal{L} , or, as the first step, deciding which formulas are negatable in \mathcal{L} . As we already remarked, even if we have established in Theorem 3.2 a characterization for negatable formulas, the latter problem is undecidable. Nevertheless, it is possible to identify some interesting classes of negatable formulas. This is what we will pursue in the next section.

Let us end this section by providing a translation of the weak negation $\sim \phi$ of every first-order formula in \mathcal{I} . The weak negation of any negatable formula in \mathcal{I} can be computed by going through the Σ_1^1 -translation of the formula (i.e. applying Theorem 2.4(ii) and (iv)), but since the Σ_1^1 -translation creates a number of dummy symbols (see [24] and [6]), this is not an efficient algorithm. We now give a direct translation of the weak negation of every first-order formula in \mathcal{I} .

Proposition 4.4. *If ϕ is a first-order formula, then $\sim \phi(\vec{x}) \equiv \exists \vec{w}(\vec{w} \subseteq \vec{x} \wedge \neg \phi(\vec{w}))$.*

Proof. For all models M and all teams X , since ϕ is flat,

$$M \models_X \sim \phi \iff X = \emptyset \text{ or } M \not\models_X \phi \iff X = \emptyset \text{ or } \exists s \in X (M \not\models_{\{s\}} \phi(\vec{x})).$$

By the empty team property of independence logic, it suffices to show that

$$\exists s \in X (M \not\models_{\{s\}} \phi(\vec{x})) \iff M \models_X \exists \vec{w}(\vec{w} \subseteq \vec{x} \wedge \neg \phi(\vec{w})).$$

for all models M and all nonempty teams X .

“ \implies ”: Assume $M \not\models_{\{s\}} \phi(\vec{x})$ for some $s \in X$ and $\vec{x} = x_1 \dots x_n$. For each $1 \leq i \leq n$, inductively define a constant function F_i as follows:

- $F_1 : X \rightarrow \wp(M) \setminus \{\emptyset\}$ is defined as $F_1(t) = \{s(x_1)\}$;
- $F_i : X[F_1/w_1, \dots, F_{i-1}/w_{i-1}] \rightarrow \wp(M) \setminus \{\emptyset\}$ is defined as $F_i(t) = \{s(x_i)\}$.

Consider the team $X[\vec{F}/\vec{w}]$ (see Figure 1 for an example of such a team). Clearly, $M \models_{X[\vec{F}/\vec{w}]} \vec{w} \subseteq \vec{x}$. On the other hand, for any $t \in X[\vec{F}/\vec{w}]$, since $t(\vec{w}) = s(\vec{x})$ and $M \not\models_{\{s\}} \phi(\vec{x})$, we obtain $M \not\models_{\{t\}} \phi(\vec{w})$ by the locality property. Since $\phi(\vec{w})$ is a first-order formula, the Law of Excluded Middle holds. Thus we obtain $M \models_{\{t\}} \neg \phi(\vec{w})$, which yields $M \models_{X[\vec{F}/\vec{w}]} \neg \phi(\vec{w})$ by the flatness property.

“ \impliedby ”: Conversely, suppose $M \models_X \exists \vec{w}(\vec{w} \subseteq \vec{x} \wedge \neg \phi(\vec{w}))$. Then there are appropriate functions F_i for each $1 \leq i \leq n$ such that $M \models_{X[\vec{F}/\vec{w}]} \vec{w} \subseteq \vec{x}$ and $M \models_{X[\vec{F}/\vec{w}]} \neg \phi(\vec{w})$. By the downward closure property, the latter implies that $M \not\models_{\{t\}} \phi(\vec{w})$ for some $t \in X[\vec{F}/\vec{w}]$. By the former, there exists $s' \in X[\vec{F}/\vec{w}]$ such that $s'(\vec{x}) = t(\vec{w})$. This means, by the definition of $X[\vec{F}/\vec{w}]$, that there exists $s \in X$ such that $s(\vec{x}) = s'(\vec{x}) = t(\vec{w})$. Hence, $M \not\models_{\{s\}} \phi(\vec{x})$ by the locality property. \square

A team X				A team $X[F/\vec{w}]$					
	x_1	\dots	x_n	x_1	\dots	x_n	w_1	\dots	w_n
	a_1	\dots	a_n	a_1	\dots	a_n	b_1	\dots	b_n
s	b_1	\dots	b_n	b_1	\dots	b_n	b_1	\dots	b_n
	c_1	\dots	c_n	c_1	\dots	c_n	b_1	\dots	b_n

Figure 1

5. A hierarchy of negatable atoms

In this section, we define an interesting class of formulas that are negatable in \mathcal{I} . This class will be presented in the form of an alternating hierarchy of atoms that are definable in \mathcal{I} . These atoms are closely related to the *dependency notions* considered in [7], and the *generalized dependence atoms* studied in [22] and [17]. We will demonstrate that all first-order formulas, dependence atoms, independence atoms and inclusion atoms belong to this class. At the end of the section, we will also show that the set of negatable formulas is closed under Boolean connectives and weak quantifiers. Therefore formulas in the Boolean and weak quantifier closure of the set of atoms from the hierarchy are all negatable. It then follows from the completeness result we obtained in the previous section that consequences in \mathcal{I} of these types are derivable in the extended system.

Let us start by defining the notion of abstract relation. A k -ary *relation* R is a class of pairs (M, R^M) that is closed under taking isomorphic images, where M ranges over first-order models and $R^M \subseteq M^k$. For instance, the familiar equality $=$ is a binary relation defined by the class

$$\{(M, =^M) \mid M \text{ is a first-order model}\}, \text{ where } =^M := \{(a, a) \mid a \in M\}.$$

Every first-order formula $\phi(x_1, \dots, x_k)$ with k free variables is associated with a k -ary relation

$$\phi := \{(M, \phi^M) \mid M \text{ is a first-order model}\},$$

where $\phi^M := \{(s(x_1), \dots, s(x_k)) \mid M \models_s \phi\}$. A k -ary relation R is said to be (*first-order*) *definable* if there exists a (first-order) formula $\phi_R(w_1, \dots, w_k)$ such that for all models M and all assignments s ,

$$s(\vec{w}) \in R^M \iff M \models_s \phi_R(\vec{w}).$$

Clearly, the first-order formula $w = u$ defines the equality relation, and every first-order formula ϕ defines its associated relation ϕ .

If R is a k -ary relation, then we write \bar{R} for the complement of R that is defined by letting $\bar{R}^M = M^k \setminus R^M$ for all models M . Clearly, if a first-order formula ϕ defines R , then its negation $\neg\phi$ defines \bar{R} .

If $\vec{s} = \langle s_1, \dots, s_k \rangle$, then we write $\vec{s}(\vec{x})$ for $\langle s_1(\vec{x}), \dots, s_k(\vec{x}) \rangle$. For every sequence $k = \langle k_1, \dots, k_n \rangle$ of natural numbers and every $(k_1 + \dots + k_n)m$ -ary relation R , we introduce two new atomic formulas $\Sigma_{n,k}^R(x_1, \dots, x_m)$ and $\Pi_{n,k}^R(x_1, \dots, x_m)$ with the semantics defined as follows:

- $M \models_{\emptyset} \Sigma_{n,k}^R(\vec{x})$ and $M \models_{\emptyset} \Pi_{n,k}^R(\vec{x})$.
- If n is odd, then define for any model M and any *nonempty* team X
 - $M \models_X \Sigma_{n,k}^R(\vec{x})$ iff there exist $s_{11}, \dots, s_{1k_1} \in X$ such that for all $s_{21}, \dots, s_{2k_2} \in X, \dots$ there exist $s_{n1}, \dots, s_{nk_n} \in X$ such that $(\vec{s}_1(\vec{x}), \dots, \vec{s}_n(\vec{x})) \in R^M$;
 - $M \models_X \Pi_{n,k}^R(\vec{x})$ iff for all $s_{11}, \dots, s_{1k_1} \in X$, there exist $s_{21}, \dots, s_{2k_2} \in X$ such that ... for all $s_{n1}, \dots, s_{nk_n} \in X$, it holds that $(\vec{s}_1(\vec{x}), \dots, \vec{s}_n(\vec{x})) \in R^M$.
- Similarly if n is even.

Fact 5.1. $\sim \Sigma_{n,k}^R(\vec{x}) \equiv \Pi_{n,k}^{\overline{R}}(\vec{x})$ and $\sim \Pi_{n,k}^R(\vec{x}) \equiv \Sigma_{n,k}^{\overline{R}}(\vec{x})$.

Let us now give some examples of the $\Sigma_{n,k}^R$ and $\Pi_{n,k}^R$ atoms.

Example 5.2. (a) The dependence atom $\text{dep}_k(x_1, \dots, x_k, y)$ is a $\Pi_{1,\langle 2 \rangle}^{\text{dep}_k}(x_1, \dots, x_k, y)$ atom, where dep_k is a $2(k+1)$ -ary relation defined as

$$(a_1, \dots, a_k, b, a'_1, \dots, a'_k, b') \in (\text{dep}_k)^M \text{ iff } [(a_1, \dots, a_k) = (a'_1, \dots, a'_k) \implies b = b'].$$

The first-order formula $((w_1 = w'_1) \wedge \dots \wedge (w_k = w'_k)) \rightarrow (u = u')$ defines dep_k .

(b) The independence atom $x_1, \dots, x_k \perp_{z_1, \dots, z_n} y_1, \dots, y_m$ is a

$$\Pi_{2,\langle 2,1 \rangle}^{\text{ind}_{k,m,n}}(x_1, \dots, x_k, y_1, \dots, y_m, z_1, \dots, z_n)$$

atom, where $\text{ind}_{k,m,n}$ is a (first-order definable) $(2+1)(k+m+n)$ -ary relation defined as $(\vec{a}, \vec{b}, \vec{c}, \vec{a}', \vec{b}', \vec{c}', \vec{a}'', \vec{b}'', \vec{c}'') \in (\text{ind}_{k,m,n})^M$ iff

$$(c_n, \dots, c_n) = (c'_1, \dots, c'_n) = (c''_1, \dots, c''_n) \\ \implies [(a''_1, \dots, a''_k) = (a_1, \dots, a_k) \text{ and } (b''_1, \dots, b''_m) = (b'_1, \dots, b'_m)].$$

(c) The inclusion atom $x_1, \dots, x_k \subseteq y_1, \dots, y_k$ is a $\Pi_{2,\langle 1,1 \rangle}^{\text{inc}_k}(x_1, \dots, x_k, y_1, \dots, y_k)$ atom, where inc_k is a (first-order definable) $(1+1)2k$ -ary relation defined as

$$(a_1, \dots, a_k, b_1, \dots, b_k, a'_1, \dots, a'_k, b'_1, \dots, b'_k) \in (\text{inc}_k)^M \text{ iff } (a_1, \dots, a_k) = (b'_1, \dots, b'_k).$$

(d) Every first-order formula $\phi(x_1, \dots, x_k)$ is a $\Pi_{1,\langle 1 \rangle}^{\phi}(x_1, \dots, x_k)$ atom, where ϕ is a (first-order definable) $1 \cdot k$ -ary relation defined as

$$(a_1, \dots, a_k) \in \phi^M \text{ iff } M \models_{s_{\vec{a}}} \phi \text{ where } s_{\vec{a}}(x_i) = a_i \text{ for all } i.$$

In what follows, let $\mathbf{k} = \langle k_1, \dots, k_n \rangle$ be an arbitrary sequence of natural numbers, $\vec{x} = \langle x_1, \dots, x_m \rangle$ an arbitrary sequence of variables, and R an arbitrary $(k_1 + \dots + k_n)m$ -ary relation. Suppose R is definable by a formula $\phi_R(\vec{w}_1, \dots, \vec{w}_n)$, where $\vec{w}_i = \langle w_{i,1}, \dots, w_{i,k_i} \rangle$ and $w_{i,j} = \langle w_{i,j,1}, \dots, w_{i,j,m} \rangle$. The $\Sigma_{n,k}^R(\vec{x})$ and $\Pi_{n,k}^R(\vec{x})$ atoms can be translated into second-order logic in the same manner as in Theorem 2.4. For instance, if n is even, let S be a fresh m -ary relation symbol and let $\tau_{\Sigma_{n,k}^R(\vec{x})}(S) :=$

$$\exists \vec{w}_1 \left(S(w_{1,1}) \wedge \cdots \wedge S(w_{1,k_1}) \wedge \forall \vec{w}_2 \left(S(w_{2,1}) \wedge \cdots \wedge S(w_{2,k_2}) \rightarrow \exists \vec{w}_3 \cdots \right. \right. \\ \left. \left. \cdots \exists \vec{w}_n \left(S(w_{n,1}) \wedge \cdots \wedge S(w_{n,k_n}) \wedge \phi_R(\vec{w}_1, \dots, \vec{w}_n) \right) \cdots \right) \right).$$

Then, we have $M \models_X \Sigma_{n,k}^R(\vec{x}) \iff (M, \text{rel}(X)) \models \tau_{\Sigma_{n,k}^R(\vec{x})}(S)$ for any model M and any team X . If $\phi_R(\vec{w}_1, \dots, \vec{w}_n)$ is a first-order formula, i.e., if R is first-order definable, then $\tau_{\Sigma_{n,k}^R(\vec{x})}(S)$ is a first-order sentence. This shows, by Theorem 3.2(i), that $\Sigma_{n,k}^R(\vec{x})$ and $\Pi_{n,k}^R(\vec{x})$ atoms are negatable in \mathcal{I} as long as R is first-order definable.

Yet, in order to apply the rules of the extended deduction system defined in Section 4 to derive the $\Sigma_{n,k}^R(\vec{x})$ and $\Pi_{n,k}^R(\vec{x})$ consequences in \mathcal{I} , one needs to compute the formulas that are equivalent to the weak classical negations of the $\Sigma_{n,k}^R(\vec{x})$ and $\Pi_{n,k}^R(\vec{x})$ atoms in the original language of \mathcal{I} . This can be done by applying 5.1 and going through the inefficient Σ_1^1 -translation (as we pointed out in the previous section). In what follows, we will give a direct definition of the atoms $\Sigma_{n,k}^R(\vec{x})$ and $\Pi_{n,k}^R(\vec{x})$ in the original language of \mathcal{I} .

For each $1 \leq i \leq n$, define

- $\text{inc}(w_{i,1}, \dots, w_{i,k_i}; \vec{x}) := \bigwedge_{j=1}^{k_i} (w_{i,j} \subseteq \vec{x})$
- $\text{pro}(\vec{w}_1, \dots, \vec{w}_{i-1}; \vec{x}; w_{i,1}, \dots, w_{i,k_i}) :=$

$$\left(\bigwedge_{j=1}^{k_i} (\vec{x} \subseteq w_{i,j}) \right) \wedge \left(\bigwedge_{j=1}^{k_i} (\langle w_{i,j'} \mid j' \neq j \rangle \perp w_{i,j}) \right) \wedge (\vec{w}_1 \dots \vec{w}_{i-1} \perp w_{i,1} \dots w_{i,k_i})$$

and define inductively formulas σ_i and π_i as follows:

- $\sigma_1[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)] := \exists \vec{w}_n \left(\text{inc}(w_{n,1}, \dots, w_{n,k_n}; \vec{x}) \wedge \phi_R(\vec{w}_1, \dots, \vec{w}_n) \right)$
- $\pi_1[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)] := \exists \vec{w}_n \left(\text{pro}(\vec{w}_1, \dots, \vec{w}_{n-1}; \vec{x}; w_{n,1}, \dots, w_{n,k_n}) \wedge \phi_R(\vec{w}_1, \dots, \vec{w}_n) \right)$
- $\sigma_{i+1}[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)] := \exists \vec{w}_{n-i} \left(\text{inc}(w_{n-i,1}, \dots, w_{n-i,k_{n-i}}; \vec{x}) \right. \\ \left. \wedge \pi_i[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)] \right)$
- $\pi_{i+1}[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)] := \exists \vec{w}_{n-i} \left(\text{pro}(\vec{w}_1, \dots, \vec{w}_{n-i-1}; \vec{x}; w_{n-i,1}, \dots, w_{n-i,k_{n-i}}) \right. \\ \left. \wedge \sigma_i[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)] \right)^2$

Theorem 5.3. *Let R and ϕ_R be as above. Then*

- $\Sigma_{n,k}^R(x_1, \dots, x_m) \equiv \sigma_n[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)]$

²If $i+1 = n$, then $\vec{w}_1 \dots \vec{w}_{n-i-1}$ denotes the empty sequence $\langle \rangle$ and we stipulate $\langle \rangle \perp \vec{y} := \top$.

A team X	A team $X[\overline{F_1}/\overline{w_1}]$	A team $X[\overline{F_1}/\overline{w_1}, \overline{F_2}/\overline{w_2}]$																																																																																				
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Figure 2

- $\Pi_{n,k}^R(x_1, \dots, x_m) \equiv \pi_n[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)]$

To prove the above theorem, we will need the following lemma.

Lemma 5.4. *Let X be a nonempty team of a model M with $x_1, \dots, x_m \in \text{dom}(X)$.*

- (i) *Let $\gamma : X \rightarrow X$ be a choice function. Define inductively functions F_1, \dots, F_m to simulate assignments in $\gamma[X]$ restricted to \vec{x} on a sequence $\vec{w} = \langle w_1, \dots, w_m \rangle$ of new variables as follows:*

- *Define the function $F_1 : X \rightarrow \wp(M) \setminus \{\emptyset\}$ as $F_1(t) = \{\gamma(t)(x_1)\}$.*
- *For each $2 \leq i \leq m$, define the function $F_i : X[F_1/w_1, \dots, F_{i-1}/w_{i-1}] \rightarrow \wp(M) \setminus \{\emptyset\}$ as $F_i(t) = \{\gamma(t)(x_i)\}$.*

We call $\vec{F} = \langle F_1, \dots, F_m \rangle$ the sequence of simulating functions for $\gamma[X] \upharpoonright \vec{x}$ on \vec{w} . Let $Y = X[\vec{F}/\vec{w}]$ (see Figure 1 for an example of such a team with a constant choice function $\gamma(t) = s$ for all $t \in X$, or Figure 2(b) for another example with an obvious choice function). Then, $t(\vec{w}) = \gamma(t)(\vec{x})$ for all $t \in Y$ and $M \models_Y \text{inc}(\vec{w}; \vec{x})$.

For a sequence $\Gamma = \langle \gamma_1, \dots, \gamma_k \rangle$ of choice functions $\gamma_i : X \rightarrow X$,

- *let \vec{F}_1 be the sequence of simulating functions for $\gamma_1[X] \upharpoonright \vec{x}$ on \vec{w}_1 ,*
- *and for each $2 \leq i \leq k$, let \vec{F}_i be the sequence of simulating functions for $\gamma_i[X[\vec{F}_1/\vec{w}_1, \dots, \vec{F}_{i-1}/\vec{w}_{i-1}]] \upharpoonright \vec{x}$ on \vec{w}_i .*

We call $\vec{F}_1, \dots, \vec{F}_k$ the group of simulating functions for $\Gamma[X] \upharpoonright \vec{x}$ on $\vec{w}_1, \dots, \vec{w}_k$, and the team $Y = X[\vec{F}_1/\vec{w}_1, \dots, \vec{F}_k/\vec{w}_k]$ its associated team (see Figure 2 in Appendix II for examples of such teams). Then, $M \models_Y \text{inc}(\vec{w}_1, \dots, \vec{w}_k; \vec{x})$.

A team X			A team $X[\vec{F}_1/\vec{w}_1]$				A team $X[\vec{F}_1/\vec{w}_1, \vec{F}_2/\vec{w}_2]$				
	\vec{x}	\vec{y}		\vec{x}	\vec{y}	\vec{w}_1		\vec{x}	\vec{y}	\vec{w}_1	\vec{w}_2
s_1	\vec{a}	\vec{b}	s_{11}	\vec{a}	\vec{b}	\vec{a}	s_{111}	\vec{a}	\vec{b}	\vec{a}	\vec{a}
s_2	\vec{b}	\vec{a}	s_{12}	\vec{a}	\vec{b}	\vec{b}	s_{112}	\vec{a}	\vec{b}	\vec{a}	\vec{b}
			s_{21}	\vec{b}	\vec{a}	\vec{a}	s_{121}	\vec{a}	\vec{b}	\vec{b}	\vec{a}
			s_{22}	\vec{b}	\vec{a}	\vec{b}	s_{122}	\vec{a}	\vec{b}	\vec{b}	\vec{b}
							s_{211}	\vec{b}	\vec{a}	\vec{a}	\vec{a}
							s_{212}	\vec{b}	\vec{a}	\vec{a}	\vec{b}
							s_{221}	\vec{b}	\vec{a}	\vec{b}	\vec{a}
							s_{222}	\vec{b}	\vec{a}	\vec{b}	\vec{b}

(a)

(b)

(c)

Figure 3

(ii) Define inductively functions F_1, \dots, F_m to duplicate assignments in X restricted to \vec{x} on a sequence $\vec{w} = \langle w_1, \dots, w_m \rangle$ of new variables as follows:

- Define the function $F_1 : X \rightarrow \wp(M) \setminus \{\emptyset\}$ as $F_1(t) = \{s(x_1) \mid s \in X\}$.
- For each $2 \leq i \leq m$, define the function $F_i : X[F_1/w_1, \dots, F_{i-1}/w_{i-1}] \rightarrow \wp(M) \setminus \{\emptyset\}$ as

$$F_i(t) = \{s(x_i) \mid s \in X \text{ and } s \upharpoonright \{x_1, \dots, x_{i-1}\} = t \upharpoonright \{w_1, \dots, w_{i-1}\}\}.$$

We call $\vec{F} = \langle F_1, \dots, F_m \rangle$ the sequence of duplicating functions for $X \upharpoonright \vec{x}$ on \vec{w} . (see Figure 3(b) for an example of a team $X[\vec{F}/\vec{w}]$).

For a team X ,

- let \vec{F}_1 be the sequence of duplicating functions for $X \upharpoonright \vec{x}$ on \vec{w}_1 ,
- and for each $i = 2, \dots, k$, let \vec{F}_i be the sequence of duplicating functions for $X[\vec{F}_1/\vec{w}_1, \dots, \vec{F}_{i-1}/\vec{w}_{i-1}] \upharpoonright \vec{x}$ on \vec{w}_i .

We call $\vec{F}_1, \dots, \vec{F}_k$ the group of duplicating functions for $X \upharpoonright \vec{x}$ on $\vec{w}_1, \dots, \vec{w}_k$. and the team $Y = X[\vec{F}_1/\vec{w}_1, \dots, \vec{F}_k/\vec{w}_k]$ its associated team (see Figure 3 for examples of such teams). Then, $M \models_Y \text{pro}(\vec{y}; \vec{x}; \vec{w}_1, \dots, \vec{w}_k)$ for any sequence \vec{y} of variables in $\text{dom}(X)$ that has no variable in common with \vec{x} and $\vec{w}_1, \dots, \vec{w}_k$, and for any $t \in Y$, there exist $s_1, \dots, s_k \in X$ such that $s_1(\vec{x}) = t(\vec{w}_1), \dots, s_k(\vec{x}) = t(\vec{w}_k)$.

Proof. We only give the detailed proof for $M \models_Y \text{pro}(\vec{y}; \vec{x}; \vec{w}_1, \dots, \vec{w}_k)$ in item (ii), i.e.,

$$M \models_Y \bigwedge_{i=1}^k (\vec{x} \subseteq \vec{w}_i) \wedge \left(\bigwedge_{i=1}^k (\langle \vec{w}_j \mid j \neq i \rangle \perp \vec{w}_i) \right) \wedge (\vec{y} \perp \vec{w}_1 \dots \vec{w}_k) \quad (6)$$

To show that Y satisfies the first conjunct of the formula in (6), it suffices to show that $M \models_{Y_i} \vec{x} \subseteq \vec{w}_i$ for each $1 \leq i \leq k$ and $Y_i = X[\vec{F}_1/\vec{w}_1, \dots, \vec{F}_i/\vec{w}_i]$.

For any $t \in Y_i$, by the definition of $Y_i = Y_{i-1}[\vec{F}_i/\vec{w}_i]$, there exists $s \in X$ such that $s(\vec{x}) = t(\vec{x})$, and

$$t' = s \cup \{(w_{i,1}, s(x_1)), \dots, (w_{i,m}, s(x_m))\} \in Y_{i-1}[F_{i,1}/w_{i,1}, \dots, F_{i,m}/w_{i,m}]$$

Thus, $t'(\vec{w}_i) = s(\vec{x}) = t(\vec{x})$, as required.

To prove that Y satisfies the second and the third conjuncts of the formula in (6), we prove a more general property that $M \models_Y \vec{w}_{i_1} \dots \vec{w}_{i_a} \perp \vec{w}_{j_1} \dots \vec{w}_{j_b} v_1 \dots v_c$ holds for any disjoint subsequences $\vec{w}_{i_1} \dots \vec{w}_{i_a}$ and $\vec{w}_{j_1} \dots \vec{w}_{j_b}$ of $\vec{w}_1 \dots \vec{w}_k$ and any variables $v_1 \dots v_c \in \text{dom}(X)$. Assume that $\{\vec{w}_{i_1} \dots \vec{w}_{i_a}, \vec{w}_{j_1} \dots \vec{w}_{j_b}\} = \{\vec{w}_{l_1} \dots \vec{w}_{l_d}\}$ with $l_1 < \dots < l_d$.

Let $s, s' \in Y$ be arbitrary. We need to find an $s'' \in Y$ such that $s''(\vec{w}_{i_1} \dots \vec{w}_{i_a}) = s(\vec{w}_{i_1} \dots \vec{w}_{i_a})$ and $s''(\vec{w}_{j_1} \dots \vec{w}_{j_b} v_1 \dots v_c) = s'(\vec{w}_{j_1} \dots \vec{w}_{j_b} v_1 \dots v_c)$. Let f be a function satisfying

$$f(\vec{w}_{l_\xi}) = \begin{cases} s(\vec{w}_{l_\xi}) & \text{if } l_\xi \in \{i_1, \dots, i_a\} \\ s'(\vec{w}_{l_\xi}) & \text{if } l_\xi \in \{j_1, \dots, j_b\} \end{cases}$$

There exists $s_1 \in X$ such that $s_1(\vec{x}) = f(\vec{w}_{l_1})$. Put $Y_{l_1-1} = X[\vec{F}_1/\vec{w}_1, \dots, \vec{F}_{l_1-1}/\vec{w}_{l_1-1}]$ and $t = s' \upharpoonright \text{dom}(Y_{l_1-1})$. By the construction,

$$t_{l_1} = t \cup \{(w_{l_1,1}, s_1(x_1)), \dots, (w_{l_1,m}, s_1(x_m))\} \in Y_{l_1-1}[\vec{F}_{l_1}/\vec{w}_{l_1}] = Y_{l_1}.$$

Thus

$$t_{l_1}(\vec{w}_{l_1}) = s_1(\vec{x}) = f(\vec{w}_{l_1}) \text{ and } t_{l_1}(\vec{v}) = t(\vec{v}) = s'(\vec{v}).$$

Repeat the same argument for $f(\vec{w}_{l_2}), \dots, f(\vec{w}_{l_d})$, we can find $t_{l_d} \in Y_{l_d}$ such that

$$t_{l_d}(\vec{w}_{i_1} \dots \vec{w}_{i_a}) = s(\vec{w}_{i_1} \dots \vec{w}_{i_a}) \text{ and } t_{l_d}(\vec{w}_{j_1} \dots \vec{w}_{j_b} v_1 \dots v_c) = s'(\vec{w}_{j_1} \dots \vec{w}_{j_b} v_1 \dots v_c).$$

Finally, by the construction of Y , there exists $s'' \in Y$ such that $s'' \upharpoonright \text{dom}(Y_{l_d}) = t_{l_d}$. Hence, s'' is the desired assignment. \square

Now, we are ready to prove Theorem 5.3.

Proof of Theorem 5.3. We only give the detailed proof for $\Sigma_{n,k}^R(x_1, \dots, x_m)$ when n is odd. The other case and the other equivalence can be proved analogously.

First, note that

$$\begin{aligned} \sigma_n[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)] &:= \exists \vec{w}_1 \left(\text{inc}(\vec{w}_{1,1}, \dots, \vec{w}_{1,k_1}; \vec{x}) \wedge \right. \\ &\quad \exists \vec{w}_2 \left(\text{pro}(\vec{w}_1; \vec{x}; \vec{w}_{2,1}, \dots, \vec{w}_{2,k_2}) \wedge \dots \wedge \right. \\ &\quad \left. \left. \exists \vec{w}_n \left(\text{inc}(\vec{w}_{n,1}, \dots, \vec{w}_{n,k_n}; \vec{x}) \wedge \phi_R(\vec{w}_1, \dots, \vec{w}_n) \right) \dots \right) \right) \end{aligned}$$

$\underbrace{\hspace{10em}}_n$

Suppose $M \models_X \Sigma_{n,k}^R(\vec{x})$ for some model M and some nonempty team X . Then

$$(\exists \vec{s}_1 \in X^{k_1})(\exists \vec{s}_2 \in X^{k_2}) \dots (\exists \vec{s}_n \in X^{k_n})(\vec{s}_1(\vec{x}), \dots, \vec{s}_n(\vec{x})) \in R^M. \quad (7)$$

Let $\Gamma_1 = \langle \gamma_{1,1}, \dots, \gamma_{1,k_1} \rangle$ be a sequence of constant choice functions $\gamma_{1,j} : X \rightarrow X$ defined as $\gamma_{1,j}(t) = s_{1,j}$. Let $\vec{F}_{1,1}, \dots, \vec{F}_{1,k_1}$ be the group of simulating functions for $\Gamma_1[X] \upharpoonright \vec{x}$ on $w_{1,1}, \dots, w_{1,k_1}$ and Y_1 its associated team defined as in 5.4(i) in Appendix III. Then, $M \models_{Y_1} \text{inc}(w_{1,1}, \dots, w_{1,k_1}; \vec{x})$. It then remains to show that $M \models_{Y_1} \pi_{n-1}[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)]$.

Let $\vec{F}_{2,1}, \dots, \vec{F}_{2,k_2}$ be the group of duplicating functions for $Y_1 \upharpoonright \vec{x}$ on $w_{2,1}, \dots, w_{2,k_2}$, and Y_2 its associated team defined as in 5.4(ii) in Appendix III. Then, $M \models_{Y_2} \text{pro}(\vec{w}_1; \vec{x}; w_{2,1}, \dots, w_{2,k_2})$.

It remains to show that $M \models_{Y_2} \sigma_{n-2}[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)]$. By 5.4(ii), for each $t \in Y_2$, there exists $\vec{s}_{t,2} = (s_{t,1}^t, \dots, s_{t,k_2}^t) \in X^{k_2}$ satisfying

$$s_{t,1}^t(\vec{x}) = t(w_{2,1}), \dots, s_{t,k_2}^t(\vec{x}) = t(w_{2,k_2}).$$

Hence, by (7), there exists $\vec{s}_{t,3} = (s_{t,1}^t, \dots, s_{t,k_3}^t) \in X^{k_3}$ such that

$$(\forall \vec{s}_4 \in X^{k_4}) \dots (\exists \vec{s}_n \in X^{k_n})(\vec{s}_1(\vec{x}), \vec{s}_{t,2}(\vec{x}), \vec{s}_{t,3}(\vec{x}), \vec{s}_4(\vec{x}), \dots, \vec{s}_n(\vec{x})) \in R^M.$$

Let $\Gamma_3 = \langle \gamma_{3,1}, \dots, \gamma_{3,k_3} \rangle$ be a sequence of choice functions $\gamma_{3,j} : Y_2 \rightarrow Y_2$ defined as $\gamma_{3,j}(t) = s_{t,j}^t$. Let $\vec{F}_{3,1}, \dots, \vec{F}_{3,k_3}$ be the group of simulating functions for $\Gamma_3[Y_2] \upharpoonright \vec{x}$ on $w_{3,1}, \dots, w_{3,k_3}$, and Y_3 its associated team defined as in 5.4(i) in Appendix III. Then, $M \models_{Y_3} \text{inc}(w_{3,1}, \dots, w_{3,k_3}; \vec{x})$ and it remains to show that $M \models_{Y_3} \pi_{n-3}[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)]$.

Repeat the argument n times. In the last step we have Y_n and Γ_n defined and $M \models_{Y_n} \text{inc}(w_{n,1}, \dots, w_{n,k_n}; \vec{x})$ by 5.4(i). It then only remains to show that $M \models_{Y_n} \phi_R(\vec{w}_1, \dots, \vec{w}_n)$. Since ϕ_R is flat, it suffices to show that $M \models_{\{t\}} \phi_R$ holds for all $t \in Y_n$. By the definition of Y_n and 5.4(i)(ii), we have

$$(\vec{s}_1(\vec{x}), \vec{s}_{t,2}(\vec{x}), \vec{s}_{t,3}(\vec{x}), \vec{s}_{t,4}(\vec{x}), \dots, \vec{s}_{t,n}(\vec{x})) \in R^M$$

$$\text{and } t(\vec{w}_1) = \vec{s}_1(\vec{x}), t(\vec{w}_2) = \vec{s}_{t,2}(\vec{x}), \dots, t(\vec{w}_n) = \vec{s}_{t,n}(\vec{x}).$$

Thus, $M \models_{\{t\}} \phi_R(\vec{w}_1, \dots, \vec{w}_n)$, as the first-order formula ϕ_R defines R .

Conversely, suppose $M \models_X \sigma_n[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)]$ for some model M and some nonempty team X . Let Y be a team generated by the formula $\sigma_n[\vec{x}; \phi_R(\vec{w}_1, \dots, \vec{w}_n)]$ from X such that $M \models_Y \phi_R(\vec{w}_1, \dots, \vec{w}_n)$.

Pick any $t \in Y$. Since $M \models_Y \text{inc}(w_{1,1}, \dots, w_{1,k_1}; \vec{x})$, there exist $s_{1,1}, \dots, s_{1,k_1} \in X$ such that

$$s_{1,1}(\vec{x}) = t(w_{1,1}), \dots, s_{1,k_1}(\vec{x}) = t(w_{1,k_1}).$$

Let $s_{2,1}, \dots, s_{2,k_2} \in X$ be arbitrary. Since $M \models_Y \text{pro}(\vec{w}_1; \vec{x}; w_{2,1}, \dots, w_{2,k_2})$, it is not hard to see that there exist $t_2 \in Y$ such that

$$t_2(\vec{w}_1) = t(\vec{w}_1) = \vec{s}_1(\vec{x}) \text{ and } s_{2,1}(\vec{x}) = t_2(w_{2,1}), \dots, s_{2,k_2}(\vec{x}) = t_2(w_{2,k_2}).$$

Repeat the argument n times to find in the same manner the corresponding assignments $\vec{s}_3 \in X^{k_3}, \vec{s}_5 \in X^{k_5}, \dots, \vec{s}_n \in X^{k_n}$ and the corresponding assignments $t_4, t_6, \dots, t_{n-1} \in Y$ for arbitrary $\vec{s}_4 \in X^{k_4}, \vec{s}_6 \in X^{k_6}, \dots, \vec{s}_{n-1} \in X^{k_{n-1}}$. In the last step we have

$$t_{n-1}(\vec{w}_1) = \vec{s}_1(\vec{x}), \dots, t_{n-1}(\vec{w}_{n-1}) = \vec{s}_{n-1}(\vec{x})$$

and there exist $s_{n,1}, \dots, s_{n,k_n} \in X$ such that

$$s_{n,1}(\vec{x}) = t_{n-1}(\vec{w}_{n,1}), \dots, s_{n,k_n}(\vec{x}) = t_{n-1}(\vec{w}_{n,k_n}).$$

Since $M \models_Y \phi_R(\vec{w}_1, \dots, \vec{w}_n)$, we have $M \models_{\{t_{n-1}\}} \phi_R(\vec{w}_1, \dots, \vec{w}_n)$ by the downward closure property. Since the first-order formula ϕ_R defines R , we conclude

$$(t_{n-1}(\vec{w}_1), \dots, t_{n-1}(\vec{w}_n)) \in R^M \text{ yielding } (\vec{s}_1(\vec{x}), \dots, \vec{s}_n(\vec{x})) \in R^M.$$

□

Having settled a hierarchy of negatable formulas, a natural question to ask is whether it is possible to extend the hierarchy in some way while keeping the negatability of the formulas. Let us now try to answer this question.

The Σ_1^1 -translations for the logical constants the disjunction \vee , the existential quantifier \exists and universal quantifier \forall as given in [24] are as follows:

- $\tau_{\phi \vee \psi}(R) = \exists S \exists S' (\tau_\phi(S) \wedge \tau_\psi(S') \wedge \forall \vec{x} (R\vec{x} \rightarrow (S\vec{x} \vee S'\vec{x})))$
- $\tau_{\exists x \phi(x, \vec{y})}(R) = \exists S (\tau_\phi(S) \wedge \forall \vec{y} (R\vec{y} \rightarrow \exists x Sx\vec{y}))$
- $\tau_{\forall x \phi(x, \vec{y})}(R) = \exists S (\tau_\phi(S) \wedge \forall \vec{y} (R\vec{y} \rightarrow \forall x Sx\vec{y}))$

These translations can well be non-first-order sentences, and thus none of \vee , \exists and \forall preserve negatability. Nevertheless, in the literature there are other variants of these logical constants, under which the set of negatable formulas is closed. They are the disjunction $\vee\vee$ (introduced in [1]), the (weak) existential quantifier \exists^1 and the (weak) universal quantifier \forall^1 (introduced in [19]) whose semantics are defined as follows:

- $M \models_X \phi \vee\vee \psi$ iff $M \models_X \phi$ or $M \models_X \psi$
- $M \models_X \exists^1 x \phi$ iff $M \models_{X(a/x)} \phi$ for some $a \in M$
- $M \models_X \forall^1 x \phi$ iff $M \models_{X(a/x)} \phi$ for all $a \in M$

In the presence of the downward closure property the disjunction $\vee\vee$ is called *intuitionistic disjunction*, and in the environment of \mathcal{I} we shall call it *Boolean disjunction*. For all formulas ϕ of \mathcal{D} or \mathcal{I} , all models M and all teams X ,

$$M \models_X \phi \vee\vee \psi \iff (M, \text{rel}(X)) \models \tau_\phi(R) \vee \tau_\psi(R),$$

$$M \models_X \exists^1 x \phi \iff (M, \text{rel}(X)) \models \exists x \tau_\phi(R)$$

and

$$M \models_X \forall^1 x \phi \iff (M, \text{rel}(X)) \models \forall x \tau_\phi(R)$$

In view of this, Theorem 2.4 can be generalized to cover extensions of \mathcal{D} and \mathcal{I} with these three logical constants and these three logical constants are then definable in \mathcal{D} and in \mathcal{I}^3 . If $\tau_\phi(R)$ and $\tau_\psi(R)$ are both first-order, then the above three Σ_1^1 -translations $\tau_\phi(R) \vee \tau_\psi(R)$, $\exists x \tau_\phi(R)$ and $\forall x \tau_\phi(R)$ are first-order as well. By Theorem 3.3, this shows that the logical constants \vee , \exists^1 and \forall^1 preserve negatability. Furthermore, it is easy to see that

- $\sim(\phi \wedge \psi) \equiv \sim\phi \vee \sim\psi$, $\sim(\phi \vee \psi) \equiv \sim\phi \wedge \sim\psi$,
- $\sim\exists^1 x \phi \equiv \forall^1 x \sim\phi$ and $\sim\forall^1 x \phi \equiv \exists^1 x \sim\phi$.

Without going into detail we remark that making use of the Boolean disjunction \vee the extended system of \mathcal{I} can be applied to give a new formal proof of Arrow's Impossibility Theorem [3] in social choice theory. In [23] the theorem is formulated as an entailment $\Gamma_{\text{Arrow}} \models \phi_{\text{dictator}}$ in independence logic, where Γ_{Arrow} is a set of formulas expressing the conditions in Arrow's Impossibility Theorem and ϕ_{dictator} is a formula expressing the existence of a dictator. The formula ϕ_{dictator} is of the form $\vee_{i=1}^n \phi_i$, where ϕ_i is a first-order formula expressing that voter i is a dictator (among n voters). By what we just obtained, the formula ϕ_{dictator} is negatable in \mathcal{I} and the Completeness Theorem guarantees that $\Gamma_{\text{Arrow}} \vdash_{\mathcal{I}}^* \phi_{\text{dictator}}$ is derivable in our extended system.

6. Armstrong's axioms and the Geiger-Paz-Pearl axioms

Dependence and independence atoms being members of the hierarchy we have defined in the previous section (see 5.2) are negatable in independence logic. Therefore, Armstrong's Axioms [2] that characterize dependence atoms and the Geiger-Paz-Pearl axioms [9] that characterize independence atoms are derivable in our extended system of independence logic. In this section, we provide the derivations of these axioms in order to illustrate the power of our extended system.

Throughout this section we denote by \vdash the syntactic consequence relation associated with the extended system of \mathcal{I} defined in Section 4. The crucial rules from [11] that we will apply in our derivation are $\exists I$, $\exists E$ and the following ones, where we write x, y, z, \dots for arbitrary sequences of variables:

³Moreover, \exists^1 and \forall^1 are uniformly definable in \mathcal{D} and in \mathcal{I} , since $\exists^1 x \phi \equiv \exists x (=(x) \wedge \phi)$ and $\phi \vee \psi \equiv \exists w \exists u (=(w) \wedge =(u) \wedge ((w = u) \vee \phi) \wedge ((w \neq u) \vee \psi))$.

Projection and permutation of inclusion $\frac{x_1 \dots x_n \subseteq y_1 \dots y_n}{x_{i_1} \dots x_{i_k} \subseteq y_{i_1} \dots y_{i_k}} \subseteq \text{Pro}$	Transitivity of inclusion $\frac{x \subseteq y \quad y \subseteq z}{x \subseteq z} \subseteq \text{Trs}$	Inclusion compression $\frac{y \subseteq x \quad \alpha}{\alpha(y/x)} \subseteq \text{Cmp}$
Independence elimination $\frac{x \perp_z y \quad w_1 u_1 v_1 t_1 \subseteq xyzs \quad w_2 u_2 v_2 t_2 \subseteq xyzs}{\exists w_3 u_3 v_3 t_3 ((w_3 u_3 v_3 t_3 \subseteq xyzs) \wedge (v_1 = v_2 \rightarrow w_3 u_3 v_3 = w_1 u_2 v_2))} \perp E$		

Now, we derive the Geiger-Paz-Pearl axioms as follows.

Example 6.1. *The following clauses, known as the Geiger-Paz-Pearl axioms [9], are derivable in the extended system of \mathcal{I} .*

- (1) $x \perp y \vdash y \perp x$
- (2) $x \perp y \vdash z \perp y$, where z is a subsequence of x .
- (3) $x \perp y \vdash u \perp v$, where u is a permutation of x and v is a permutation of y .
- (4) $x \perp y, xy \perp z \vdash x \perp yz$.

Proof. We only give the detailed derivation for item (4). By the rule $\sim E$, it suffices to derive $x \perp y, xy \perp z, \sim(x \perp yz) \vdash \perp$, which, by the translation given in Theorem 5.3, is equivalent to

$$x \perp y, xy \perp z, \exists w_1 u_1 v_1 \exists w_2 u_2 v_2 \left((w_1 u_1 v_1 \subseteq xyz) \wedge (w_2 u_2 v_2 \subseteq xyz) \right. \\ \left. \wedge \exists w_3 u_3 v_3 ((xyz \subseteq w_3 u_3 v_3) \wedge (w_3 u_3 v_3 \neq w_1 u_2 v_2)) \right) \vdash \perp$$

By $\exists E$ and $\exists I$, the above is further equivalent to

$$x \perp y, xy \perp z, w_1 u_1 v_1 \subseteq xyz, w_2 u_2 v_2 \subseteq xyz, xyz \subseteq w_3 u_3 v_3, w_3 u_3 v_3 \neq w_1 u_2 v_2 \vdash \perp \quad (8)$$

See Figure 4 for the derivation of the above clause. \square

To state Armstrong's Axioms in full generality, we need to introduce a generalized version of dependence atoms, atoms $=(x, y)$ that may have sequence of variables in the last coordinate. The semantic clause for the atom $=(x, y)$ is

- $M \models_X =(x, y)$ iff for all $s, s' \in X$, if $s(x) = s'(x)$, then $s(y) = s'(y)$

Clearly, $=(x, y) \equiv y \perp_x y$, and we thus interpret the dependence atom $=(x, y)$ in \mathcal{I} as the independence atom $y \perp_x y$.

To derive Armstrong's Axioms we first derive some useful lemmas concerning inclusion atoms and dependence atoms in the extended system of \mathcal{I} .

Lemma 6.2. (1) $w'u'v' \subseteq wuu \vdash u' = v'$

(2) $=(w, u), s_1t_1 \subseteq wu, s_2t_2 \subseteq wu \vdash s_1 = s_2 \rightarrow t_1 = t_2$

Proof. (1) Since $\vdash u = u$, the item is proved by applying $\subseteq \text{Cmp}$.

(2) The item is derived as follows:

$$\frac{\frac{u \perp_w u \quad \frac{s_1t_1 \subseteq wu}{s_1t_1t_1 \subseteq wuu} \subseteq \text{Pro} \quad \frac{s_2t_2 \subseteq wu}{s_2t_2t_2 \subseteq wuu} \subseteq \text{Pro}}{\exists w_1u_1u_2((w_1u_1u_2 \subseteq wuu) \wedge (s_1 = s_2 \rightarrow w_1u_1u_2 = s_1t_1t_2))} \perp E}{\frac{\exists w_1u_1u_2((u_1 = u_2) \wedge (s_1 = s_2 \rightarrow w_1u_1u_2 = s_1t_1t_2))}{s_1 = s_2 \rightarrow t_1 = t_2} \text{ by item (1)}}$$

□

Example 6.3. The following clauses, known as Armstrong's Axioms [2], are derivable in the extended system of \mathcal{I} .

(1) $\vdash =(x, x)$

(2) $=(x, y, z) \vdash =(y, x, z)$

(3) $=(x, x, y) \vdash =(x, y)$

(4) $=(y, z) \vdash =(x, y, z)$

(5) $=(x, y), =(y, z) \vdash =(x, z)$

Proof. We only give the detailed derivation for item (5). By the rule $\sim E$, it suffices to derive $=(x, y), =(y, z), \sim =(x, z) \vdash \perp$, which, by the translation given in Theorem 5.3, is equivalent to

$$=(x, y), =(y, z), \exists w_1u_1v_1 \exists w_2u_2v_2 \left((w_1u_1v_1 \subseteq xyz) \wedge (w_2u_2v_2 \subseteq xyz) \wedge (w_1 = w_2) \wedge (v_1 \neq v_2) \right) \vdash \perp$$

By $\exists E$ and $\exists I$, the above is further equivalent to

$$=(x, y), =(y, z), w_1u_1v_1 \subseteq xyz, w_2u_2v_2 \subseteq xyz, w_1 = w_2, v_1 \neq v_2 \vdash \perp \quad (9)$$

We derive $=(x, y), w_1u_1v_1 \subseteq xyz, w_2u_2v_2 \subseteq xyz \vdash w_1 = w_2 \rightarrow u_1 = u_2$ as follows:

$$\frac{=(x, y) \quad \frac{w_1u_1v_1 \subseteq xyz}{w_1u_1 \subseteq xy} \subseteq \text{Pro} \quad \frac{w_2u_2v_2 \subseteq xyz}{w_2u_2 \subseteq xy} \subseteq \text{Pro}}{w_1 = w_2 \rightarrow u_1 = u_2} \text{ by 6.2(2)}$$

Similarly, we have $=(y, z), w_1u_1v_1 \subseteq xyz, w_2u_2v_2 \subseteq xyz \vdash u_1 = u_2 \rightarrow v_1 = v_2$. Hence, we conclude

$$=(x, y), =(y, z), w_1u_1v_1 \subseteq xyz, w_2u_2v_2 \subseteq xyz \vdash w_1 = w_2 \rightarrow v_1 = v_2,$$

from which (9) follows easily. □

7. Concluding remarks

In this paper, we have extended the natural deduction systems of dependence and independence logic defined in [21] and [11] and obtained complete axiomatizations of the negatable consequences in these logics. We also gave a characterization of negatable formulas in \mathcal{I} and negatable sentences in \mathcal{D} . Determining whether a formula of \mathcal{I} or \mathcal{D} is negatable is an undecidable problem. Nevertheless, we identified an interesting class of negatable formulas, the Boolean and weak quantifier closure of the class of $\Sigma_{n,k}^R$ and $\Pi_{n,k}^R$ atoms. First-order formulas, dependence and independence atoms belong to this class. We also gave derivations of Armstrong's axioms and the Geiger-Paz-Pearl axioms in our extended system of \mathcal{I} .

The results of this paper can be generalized in two directions. The first direction is to identify other negatable formulas than those in the Boolean and weak quantifier closure of the set of atoms from our hierarchy. The other direction is to analyze the $\Sigma_{n,k}^R$ and $\Pi_{n,k}^R$ atoms in more detail. As we saw in 5.2, first-order formulas and the atoms of dependence and independence situate only on the Π_1 or Π_2 level. Identifying interesting properties that situate on higher levels of the hierarchy and studying the logics that the higher level atoms induce would be an interesting topic for future research. For example, it is easy to verify that $\Pi_{1,k}^R$ atoms (including first-order formulas and dependence atoms) are closed downward, and $\Sigma_{1,k}^R$ atoms are closed upward. First-order logic extended with upward closed atoms is shown in [8] to be equivalent to first-order logic. Adding other such atoms to first-order logic results in many new logics that are expressively less than Σ_1^1 or independence logic and possibly stronger than first-order logic. These logics are potentially interesting, because, for instance, by the argument of this paper, the negatable consequences in these logics can in principle be axiomatized.

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Figure 4: The derivation of (8)

$$\begin{array}{c}
\frac{\frac{x \perp y \quad w_1 u_1 v_1 \subseteq xyz \quad w_2 u_2 v_2 \subseteq xyz}{\exists w_4 u_4 v_4 ((w_4 u_4 v_4 \subseteq xyz) \wedge (w_4 u_4 = w_1 u_2))} \perp E \quad \frac{\frac{xy \perp z \quad [w_4 u_4 v_4 \subseteq xyz]^1 \quad w_2 u_2 v_2 \subseteq xyz}{\exists w_5 u_5 v_5 ((w_5 u_5 v_5 \subseteq xyz) \wedge (w_5 u_5 v_5 = w_4 u_4 v_2))} \perp E \quad \frac{\frac{\frac{\frac{[w_5 u_5 v_5 \subseteq xyz]^2 \quad xyz \subseteq w_3 u_3 v_3}{\subseteq Trs} \quad \frac{w_5 u_5 v_5 \subseteq w_3 u_3 v_3 \quad w_3 u_3 v_3 \neq w_1 u_2 v_2}{\subseteq Cmp} \quad \frac{w_5 u_5 v_5 \neq w_1 u_2 v_2 \quad [w_5 u_5 v_5 = w_4 u_4 v_2]^2}{\perp} \quad \frac{[w_4 u_4 = w_1 u_2]^1 \quad w_4 u_4 v_2 \neq w_1 u_2 v_2}{\perp} \quad \exists E, 2}{\perp} \quad \exists E, 1}{\perp}
\end{array}$$